A correction on two dimensional KdV equation with topography*

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Abstract The correction on the 2D KdV equation derived by Djordjevic and Redekopp is presented. A lapsus calami in the 2D KdV equation is removed by means of the conservation principle of the energy flux in a wave ray tube. The results show that the coefficient of the third term in the inhomogeneous term of 2D KdV equation in the paper of Djordjevic and Redekopp is 2, instead of 3.

Keywords: 2D KdV equation, soliton, wave ray tube, correction.

Considerable attention is paid to the two dimensional KdV equation of Djordjevic and Redekopp in Ref.[1] in recent years for its application in the calculation and prediction of nonlinear internal wave fields. The obvious excellence of this equation relative to the famed K-P equation is with arbitrary topography and without the additional hypothesis in y-direction as done in the K-P equation. Unfortunately, one lapsus calami makes its application restricted due to the inclusion of a redundant factor 1 in the inhomogeneous term of the equation (2.13) in Ref. [1]. Here, a proof based on the conservation principle of energy flux is presented in order to correct the equation and to calculate three dimensional internal wave fields.

1 A general discussion on the eikonal equation of internal waves

Consider a ray tube in internal wave field, the wave energy flux in the ray tube can be defined as

$$F = L\int_0^d pq\,\mathrm{d}z, \tag{1}$$

where L is the differential width of the ray tube (the distance between neighbor rays), p the wave component of the pressure, $q=q(x, y, t)$ is the horizontal velocity along the ray, z the vertical coordinate, and d the water depth. For the continuously inhomogeneous medium, the quantity F is a conservative one. Assume l is the coordinate along the ray, (1) can be found in terms of the continuous equation and momentum equation along the ray, that is

$$\frac{\partial q}{\partial l} + \frac{\partial w}{\partial z} = 0, \tag{2}$$

$$\frac{\partial q}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial l} = 0, \tag{3}$$

and

$$w = \frac{\partial \zeta}{\partial t}. \tag{4}$$

Because all the variables $\{q, w, p, \zeta\}$ are the period processes of time t, the q and p in the integrand of (1) can be denoted by Eqs. (2)~(4), namely

$$q = c\zeta\frac{\mathrm{d}\Psi}{\mathrm{d}z}, \quad p = \rho c^2\zeta\frac{\mathrm{d}\Psi}{\mathrm{d}z}, \tag{5}$$

where $\zeta=\zeta(l, t)$ is the vertical displacement of the pycnocline, and $\Psi=\Psi(z)$ the modal function of vertical displacement. If there is not shear flow in the ocean or the background flow field is ignored, then $\Psi(z)$ is reduced to the modal function of vertical velocity $W(z)$, which is employed by Djordjevic and Redekopp[1]. Because c and L depend on the bottom profile and the ray pattern, Eqs. (1)~(5) provide the local relations of variable coefficients in the inhomogeneous term of (2.13) derived by Djordjevic and Redekopp[1].

Substituting (5) into (1), the energy flux is found as

$$F = Lc^3\Phi\zeta^2, \tag{6}$$

where

$$\Phi = \int_0^d \rho\left(\frac{\mathrm{d}\Psi}{\mathrm{d}z}\right)^2\mathrm{d}z. \tag{7}$$

Based on the conservation of the energy flux along the ray tube, it follows that

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$$\frac{\partial F}{\partial l} = 0, \tag{8}$$

that is

$$2\zeta \frac{\partial \zeta}{\partial l} L\Phi c^3 + \zeta^2 \frac{\partial}{\partial l}(L\Phi c^3) = 0, \tag{9}$$

where l is defined as the ray, and it can be found using the method of Xu[2].

From (2.5) in Ref. [1], namely

$$\tau = \epsilon^{1/2}[a(x,y) - t], \tag{10}$$

and another equivalent transformation (also see (3.1) in Ref. [1])

$$\tau = \varepsilon^{1/2}\left[\int^{l} \frac{\mathrm{d}l}{c(l)} - t\right], \quad l = l(x,y), \tag{11}$$

it follows that

$$\nabla a = \frac{\boldsymbol{l}}{c}, \tag{12}$$

where $\boldsymbol{l}$ is unit vector, c the phase velocity along the rays. Obviously, the following relation is obtained

$$(\nabla a)^2 = \frac{1}{c^2}, \tag{13}$$

where $a(x,y,t)$ is the time of the wave motion along the ray. Eq. (13) is also called the eikonal equation, and its solution gives the distribution of rays[2,3]. And it can be shown that Eq. (13) in fact is a partial differential equation of the first order, and it can be rewritten as the characteristic form, i.e. the well-known ray equations[2,3]. Once the phase velocity can be determined, the so called topography eigenvalue problem will be solved, and the disintegration of the initial internal solitons on the shelve will be determined.

For the exponential stratification, (13) reduces to the eigenvalue relation as follows (see (2.10) in the Ref. [1])

$$c^2 = \beta^{-1}\left[\frac{1}{4} + \left(\frac{n\pi}{\beta d}\right)^2\right]^{-1}. \tag{14}$$

For any stratification, (14) will be a little complicated, but the form of (13) is universal for wave field from any physical origin (surface, internal, sound), and the specificity of internal waves is on the difference of the eigenvalue problem (the relation of the phase velocity to the density stratification and the bottom profile).

2 The correction of inhomogeneous term in 2D KdV equation

Due to the lapsus calami in Eq. (2.13) of Djordjevic and Redekopp, it cannot be used to calculate the nonlinear internal wave fields. This lapsus calami leads also to some mistakes in the displacement Eq. (2.16) and Eq. (2.17). To employ the 2D KdV equation of Djordjevic and Redekopp, the correction to this equation is presented in this section.

Let $L\Phi c^3$ be equal to Λ, thus (9) can be rewritten as follows:

$$2\frac{\partial \zeta}{\partial l} + \zeta\Lambda^{-1}\frac{\partial \Lambda}{\partial l} = 0. \tag{15}$$

When the ray determined by eikonal a is taken into account, it follows that

$$\frac{\partial \zeta}{\partial l} = \boldsymbol{l} \cdot \nabla \zeta = c(a_x\zeta_x + a_y\zeta_y). \tag{16}$$

Because ζ is proportional to A [see (2.14) of Djordjevic & Redekopp], (16) becomes

$$2d\ (a_xA_x + a_yA_y) + \Re A = 0, \tag{17}$$

where

$$\Re = [d/(c\Lambda)](\mathrm{d}\Lambda/\mathrm{d}l). \tag{18}$$

Two terms in the brackets of (17) coincide with the last two terms in the first line in (2.13) of the paper of Djordjevic and Redekopp. In (17), the last term will be employed to analyze the other coefficient of the variable A, and it is expected that $\Re$ will coincide with the terms in figure brackets of 2D KdV equation (2.13) in the paper.

The differential width of the ray tube is expressed with the eikonal a. Doing the operation $\nabla\cdot$ (12) yields

$$\Delta a = \nabla \cdot \frac{\boldsymbol{l}}{c} = \frac{1}{L}\frac{\partial (L/c)}{\partial l}. \tag{19}$$

By means of (16) and (19), a simple manipulation of (18) gives

$$\Re = \nabla \cdot (d\nabla a) - \nabla a \cdot \nabla d + d\frac{L}{\Lambda c}\nabla a \cdot \nabla(\Lambda c/L). \tag{20}$$

The first term in (20) is just the first two terms of equation (2.13), i.e. $(da_x)_x + (da_y)_y$ without the use of the scale transformation. Define the remanent part in (20)

$$\wp = -\nabla a \cdot \nabla d + d\frac{L}{\Lambda c}\nabla a \cdot \nabla(\Lambda c/L), \tag{21}$$

and using the trivial manipulations, it can be rewritten as

$$\wp = \frac{Ld^2}{\Lambda c}\nabla a \cdot \nabla\left(\frac{\Lambda c}{Ld}\right). \tag{22}$$

It may be expected that $\wp$ is just the third term in the figure brackets of the first line in (2.13). It should be pointed out that the above calculations are valid for any density stratification in the ocean. In the following, the integral Φ should be found for the real stratification. In order to reduce it to the result of Djordjevic and Redekopp, the simple case of exponential stratification is considered and the Boussinesq approximation is also employed to the non-exponential term in the eigenfunction ($\beta \sim N^2 \rightarrow 0$), thus in this approximation the eigenfunction and eigenvalue are

$$\Psi = \sin\left(\frac{n\pi z}{d}\right), \quad \frac{\beta}{c^2} = \left(\frac{n\pi}{d}\right)^2. \qquad (23)$$

Through the trivial calculation, the Φ is given as follows:

$$\Phi = \frac{d}{2}\left(\frac{n\pi}{d}\right)^2, \qquad (24)$$

and in (22)

$$\frac{\Lambda c}{Ld} = \frac{\Phi c^4}{d} = \frac{\beta^2}{2}\left(\frac{d}{n\pi}\right)^2. \qquad (25)$$

Substituting (25) into (22) yields

$$\wp = d\frac{1}{d^2}\nabla d^2 \cdot \nabla a = 2\nabla a \cdot \nabla d$$
$$= 2(a_x d_x + a_y d_y). \qquad (26)$$

This is the third term in the figure brackets of (2.13) in the paper of Djordjevic and Redekopp as $\beta \rightarrow 0$ without the scale transformation.

3 Conclusion

Eq.(26) is an important result. It gives a main difference from the third term of the first line in the paper of Djordjevic and Redekopp. From the third term in the figure brackets of (2.13), it follows that

$$(a_x d_\xi + a_y d_\eta)\lim_{\beta \to 0}\left[1 + \frac{2(2n\pi/\beta d)^2}{1 + (2n\pi/\beta d)^2}\right]$$
$$= (1+2)(a_x d_\xi + a_y d_\eta) = 3(a_x d_\xi + a_y d_\eta).$$

Omiting the difference of the scale transformation, the coefficient of (26) is 2, but that of (27) derived by Djordjevic and Redekopp is 3. It is shown that there exists a redundant factor 1 in the third term of the figure brackets (see the paper of Djordjevic and Redekopp).

Based on the significant applying background of the 2D KdV equation derived by Djordjevic and Redekopp, for the lapsus calami, i.e. the redundant factor 1, it must be removed from the equation (2.13). As an original equation for the calculation and prediction of three dimensional fields of internal waves on the shelve, the proof in this paper is done based on the conservation of the energy flux in the ray tube. A direct proof on the redundant factor 1 will be published in another paper. In conclusion, it should be pointed out that the results in this paper can be applied to any stratification and bottom topography.

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